UNIT - III (Degeneracy & Optimization techniques of Transportation Problem)

Examining the Initial Basic Feasible Solution for Non-Degeneracy

Examine the initial basic feasible solution for non-degeneracy. If it is said to be non-degenerate then it has the following two properties

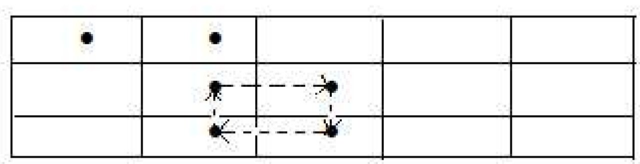
- Initial basic feasible solution must contain exactly m + n – 1 number of individual allocations.
- These allocations must be in independent positions

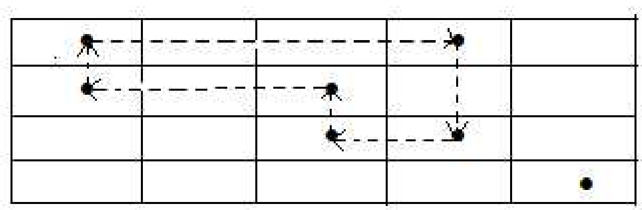
Independent Positions

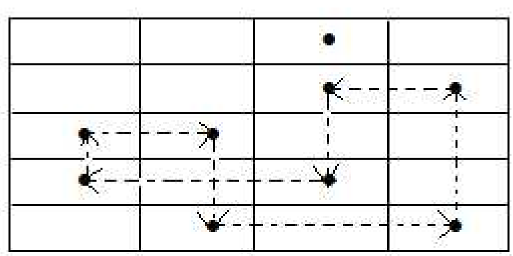
•	•	•		
		•	•	•
	•			•

•				•
			•	•
		•		•

Non-Independent Positions







Finding the Optimal Solution

Once an initial solution has been found, the next step is to test that solution for optimality. The following two methods are widely used for testing the solutions:

- **Stepping Stone Method**
- **Modified Distribution Method**

The two methods differ in their computational approach but give exactly the same results and use the same testing procedure.

Stepping-Stone Method

In this method we calculate the net cost change that can be obtained by introducing any of the unoccupied cells into the solution.

The Stepping Stone Method

1. Make sure that the number of occupied cells is exactly equal to m=n-1, where m=number of rows and n=number of columns.
2. Evaluate each unoccupied cells by following its closed path and determine its net cost change.
3. Determine the quality to be shipped to the selected unoccupied cell. Trace the closed path for the unoccupied cell and identify the minimum quality by considering the minus sign in the closed path.

Modified Distribution (MODI) Method

The MODI method is a more efficient procedure of evaluating the unoccupied cells. The modified transportation table of the initial solution is shown below

Transportation Algorithm for Minimization Problem (MODI Method)

Step 1

Construct the transportation table entering the origin capacities a_i, the destination requirement b_j and the cost c_{ij}

Step 2

Find an initial basic feasible solution by vogel's method or by any of the given method.

Step 3

For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$, calculate the values of u_i and v_j on the transportation table

Step 4

Compute the cost differences $d_{ij} = c_{ij} – (u_i + v_j)$ for all the non-basic cells

Step 5

Apply optimality test by examining the sign of each d_{ij}

- If all $d_{ij} \geq 0$, the current basic feasible solution is optimal
- If at least one $d_{ij} < 0$, select the variable x_{rs} (most negative) to enter the basis.
- Solution under test is not optimal if any d_{ij} is negative and further improvement is required by repeating the above process.

Step 6

Let the variable x_{rs} enter the basis. Allocate an unknown quantity Ө to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount Ө is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7

Assign the largest possible value to the Ө in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step 8

Now, return to step 3 and repeat the process until an optimal solution is obtained.

Worked Examples

Example 1

Find an optimal solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

1. **Applying vogel's approximation method for finding the initial basic feasible solution**

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Minimum transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2. **Check for Non-degeneracy**

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

3. **Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$**

						u_i
	(19)			(10)		$u_1 = -10$
			(40)	(60)		$u_2 = 40$
		(8)		(20)		$u_3 = 0$
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$		

Assign a 'u' value to zero. (Convenient rule is to select the u_i, which has the largest number of allocations in its row)

Let $u_3 = 0$, then

$u_3 + v_4 = 20$ which implies $0 + v_4 = 20$, so $v_4 = 20$

$u_2 + v_4 = 60$ which implies $u_2 + 20 = 60$, so $u_2 = 40$

$u_1 + v_4 = 10$ which implies $u_1 + 20 = 10$, so $u_1 = -10$

$u_2 + v_3 = 40$ which implies $40 + v_3 = 40$, so $v_3 = 0$

$u_3 + v_2 = 8$ which implies $0 + v_2 = 8$, so $v_2 = 8$

$u_1 + v_1 = 19$ which implies $-10 + v_1 = 19$, so $v_1 = 29$

4. **Calculation of cost differences for non basic cells $d_{ij} = c_{ij} - (u_i + v_j)$**

c_{ij}

	(30)	(50)	
(70)	(30)		
(40)		(70)	

$u_i + v_j$

	-2	-10	
69	48		
29		0	

$d_{ij} = c_{ij} - (u_i + v_j)$

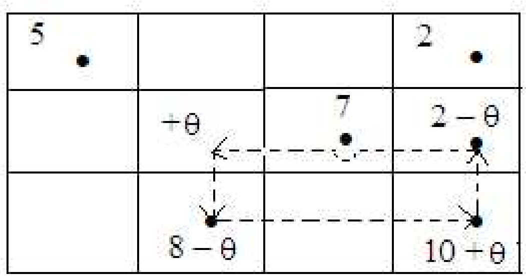
	32	60	
1	**-18**		
11		70	

5. **Optimality test**

$d_{ij} < 0$ i.e. $d_{22} = -18$

so x_{22} is entering the basis

6. **Construction of loop and allocation of unknown quantity Ө**



We allocate Θ to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Θ in the marked cell. The value of Θ is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Θ, 2-Θ) = 0 which gives Θ = 2. Therefore x_{24} is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

7. **Improved Solution**

					u_i
	(19)			(10)	$u_1 = -10$
		(30)	(40)		$u_2 = 22$
		(8)		(20)	$u_3 = 0$
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	

c_{ij}

	(30)	(50)	
(70)			(60)
(40)		(70)	

$u_i + v_j$

	-2	8	
51			42
29		18	

$d_{ij} = c_{ij} – (u_i + v_j)$

	32	42	
19			18
11		52	

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.743

Example 2

Solve by lowest cost entry method and obtain an optimal solution for the following problem

				Available
	50	30	220	1
From	90	45	170	3
	250	200	50	4
Required	4	2	2	

Solution

By lowest cost entry method

				Available
		1(30)		1/0
From	2(90)	1(45)		3/2/0
	2(250)		2(50)	4/2/0
Required	4/2/2	2/1/0	2/0	

Minimum transportation cost is 1 (30) + 2 (90) + 1 (45) + 2 (250) + 2 (50) = Rs. 855

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 3 – 1 = 5 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		(30)		$u_1 = -15$
	(90)	(45)		$u_2 = 0$
	(250)		(50)	$u_3 = 160$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -110$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

50		220
		170
	200	

$u_i + v_j$

75		-125
		-110
	205	

$d_{ij} = c_{ij} – (u_i + v_j)$

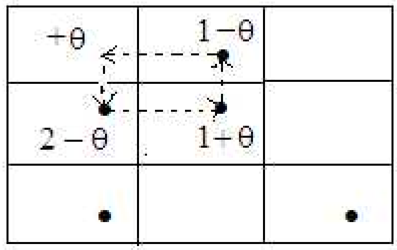
-25		345
		280
	-5	

Optimality test

$d_{ij} < 0$ i.e. $d_{11} = -25$ is most negative

So x_{11} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(2-Θ, 1-Θ) = 0 which gives Θ = 1. Therefore x_{12} is outgoing as it becomes zero.

1(50)		
1(90)	2(45)	
2(250)		2(50)

Minimum transportation cost is 1 (50) + 1 (90) + 2 (45) + 2 (250) + 2 (50) = Rs. 830

II Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	(50)			u_1= -40
	(90)	(45)		u_2 = 0
	(250)		(50)	u_3 = 160
v_j	v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

	30	220
		170
	200	

$u_i + v_j$

	5	-150
		-110
	205	

$d_{ij} = c_{ij} – (u_i + v_j)$

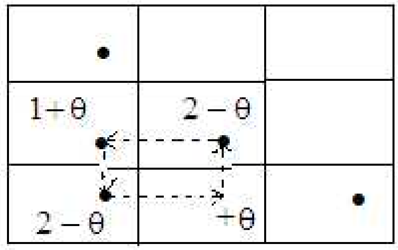
	25	370
		280
	-5	

Optimality test

$d_{ij} < 0$ i.e. d_{32} = -5

So x_{32} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



2 – ϴ = 0 which gives ϴ = 2. Therefore x_{22} and x_{31} is outgoing as it becomes zero.

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	(50)			u_1= -40
	(90)	(45)		u_2 = 0
		(200)	(50)	u_3 = 155
v_j	v_1 = 90	v_2 = 45	v_3 = -105	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

	30	220
		170
250		

$u_i + v_j$

	5	-145
		-105
245		

$d_{ij} = c_{ij} – (u_i + v_j)$

	25	365
		275
5		

Since d_{ij} > 0, an optimal solution is obtained with minimal cost Rs.820

Example 3

Is $x_{13} = 50$, $x_{14} = 20$, $x_{21} = 55$, $x_{31} = 30$, $x_{32} = 35$, $x_{34} = 25$ an optimal solution to the transportation problem.

					Available
	6	1	9	3	70
From	11	5	2	8	55
	10	12	4	7	90
Required	85	35	50	45	

Solution

					Available
			50(9)	20(3)	X
From	55(11)				X
	30(10)	35(12)		25(7)	X
Required	X	X	X	X	

Minimum transportation cost is 50 (9) + 20 (3) + 55 (11) + 30 (10) + 35 (12) + 25 (7) = Rs. 2010

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
			(9)	(3)	$u_1 = -4$
	(11)				$u_2 = 1$
	(10)	(12)		(7)	$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 12$	$v_3 = 13$	$v_4 = 7$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	1		
	5	2	8
		4	

$u_i + v_j$

6	8		
	13	14	8
		13	

$d_{ij} = c_{ij} – (u_i + v_j)$

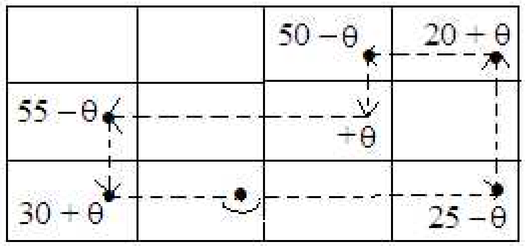
0	-7		
	-8	-12	0
		-9	

Optimality test

$d_{ij} < 0$ i.e. $d_{23} = -12$ is most negative

So x_{23} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(50-ϴ, 55-ϴ, 25-ϴ) = 25 which gives ϴ = 25. Therefore x_{34} is outgoing as it becomes zero.

		25(9)	45(3)
30(11)		25(2)	
55(10)	35(12)		

Minimum transportation cost is 25 (9) + 45 (3) + 30 (11) + 25 (2) + 55 (10) + 35 (12) = Rs. 1710

II iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
			(9)	(3)	u_1= 8
	(11)		(2)		u_2 = 1
	(10)	(12)			u_3 = 0
v_j	v_1 = 10	v_2 = 12	v_3 = 1	v_4 = -5	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	1		
	5		8
		4	7

$u_i + v_j$

18	20		
	13		-4
		1	-5

$d_{ij} = c_{ij} – (u_i + v_j)$

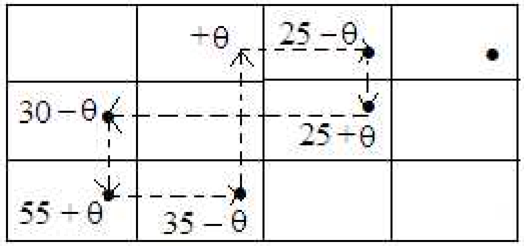
-12	-19		
	-8		12
		3	12

Optimality test

$d_{ij} < 0$ i.e. d_{12} = -19 is most negative

So x_{12} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(25-ϴ, 30-ϴ, 35-ϴ) = 25 which gives ϴ = 25. Therefore x_{13} is outgoing as it becomes zero.

	25(1)		45(3)
5(11)		50(2)	
80(10)	10(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (11) + 50 (2) + 80 (10) + 10 (12) = Rs. 1235

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		(1)		(3)	u_1= -11
	(11)		(2)		u_2 = 1
	(10)	(12)			u_3 = 0
v_j	v_1 = 10	v_2 = 12	v_3 = 1	v_4 = 14	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6		9	
	5		8
		4	7

$u_i + v_j$

-1		-10	
	13		15
		1	14

$d_{ij} = c_{ij} – (u_i + v_j)$

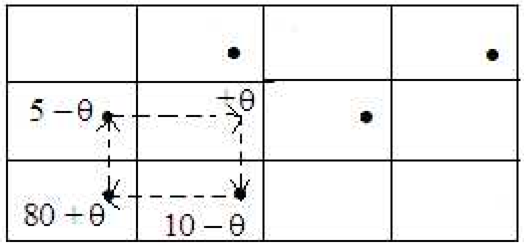
7		19	
	-8		-7
		3	-7

Optimality test

$d_{ij} < 0$ i.e. d_{22} = -8 is most negative

So x_{22} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(5-Ɵ, 10-Ɵ) = 5 which gives Ɵ = 5. Therefore x_{21} is outgoing as it becomes zero.

	25(1)		45(3)
	5(5)	50(2)	
85(10)	5(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (12) = Rs. 1195

IV Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		(1)		(3)	u_1= -11
		(5)	(2)		u_2 = -7
	(10)	(12)			u_3 = 0
v_j	v_1 = 10	v_2 = 12	v_3 = 9	v_4 = 14	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6		9	
11			8
		4	7

$u_i + v_j$

-1		-2	
3			7
		9	14

$d_{ij} = c_{ij} – (u_i + v_j)$

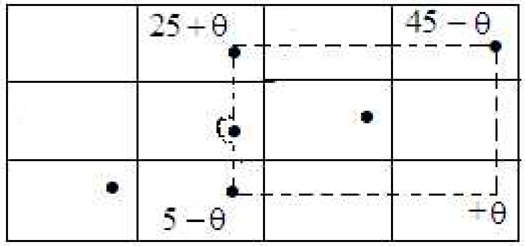
7		11	
8			1
		-5	-7

Optimality test

$d_{ij} < 0$ i.e. $d_{34} = -7$ is most negative

So x_{34} is entering the basis

Construction of loop and allocation of unknown quantity Ɵ



min(5-ϴ, 45-ϴ) = 5 which gives ϴ = 5. Therefore x_{32} is outgoing as it becomes zero.

	30(1)		40(3)
	5(5)	50(2)	
85(10)			5(7)

Minimum transportation cost is 30 (1) + 40 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (7) = Rs. 1160

V Iteration: -Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		(1)		(3)	u_1= -4
		(5)	(2)		u_2 = 0
	(10)			(7)	u_3 = 0
v_j	v_1 = 10	v_2 = 5	v_3 = 2	v_4 = 7	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6		9	
11			8
	12	4	

$u_i + v_j$

6		-2	
10			7
	5	2	

$d_{ij} = c_{ij} – (u_i + v_j)$

0		11	
1			1
	7	2	

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.1160. Further more $d_{11} = 0$ which indicates that alternative optimal solution also exists.

Exercise

1. Determine the optimal solution of the given transportation problem

	To				Supply
	2	3	11	7	6
From	1	0	6	1	1
	5	8	15	10	10
Demand	7	5	3	2	17

[Ans. $x_{12} = 5$, $x_{13} = 1$, $x_{24} = 1$, $x_{31} = 7$, $x_{33} = 2$, $x_{34} = 1$ Min cost = Rs 102]

2. Using North-West Corner rule for initial basic feasible solution, obtain an optimum basic feasible solution to the following problem

	To			Available
	7	3	4	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

[Ans. $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$ Min cost = Rs 33]

3. Determine the optimal solution of the given transportation problem

	To				Supply
	10	7	3	6	3
From	1	6	7	3	5
	7	4	5	3	7
Demand	3	2	6	4	

[Ans. $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$, $x_{33} = 3$, $x_{34} = 2$, Min cost = Rs 47]